On local solutions of the Ramanujan equation and their connection formulae

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Abstract

We show connection formulae of local solutions of the Ramanujan equation between the origin and the infinity. These solutions are given by the Ramanujan function, the q-Airy function and the divergent basic hypergeometric series ${}_2\varphi_0(0,0;-;q,x)$. We use two different q-Borel-Laplace transformations to obtain our connection formulae.

1 Introduction

In this papar, we show two essentially different connection formulae of some basic hypergeometric series between the origin and the infinity. In 1846, E. Heine [5] introduced the basic hypergeometric series ${}_2\varphi_1(a,b;c;q,x)$ as follows;

$$ {}_2\varphi_1(a,b;c;q,x) := \sum_{n\geq 0} \frac{(a,b;q)_n}{(c;q)_n(q;q)_n} x^n, \quad c \notin q^{-\mathbb{N}}. \tag{1}$$

Here, $(a;q)_n$ is the q-shifted factorial;

$$(a;q)_n := \begin{cases} 1, & n=0, \\ (1-a)(1-aq)\dots(1-aq^{n-1}), & n\geq 1, \end{cases}$$

moreover, $(a;q)_\infty := \lim_{n\to\infty}(a;q)_n$ and

$$(a_1,a_2,\dots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty\dots(a_m;q)_\infty.$$

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The q-shifted factorial $(a;q)_n$ is a q-analogue of the shifted factorial $(\alpha)_n$;

$$(\alpha)_n := \begin{cases} 1, & n = 0, \\ \alpha(\alpha+1)\dots\{\alpha+(n-1)\}, & n \geq 1. \end{cases}$$

The basic hypergeometric series (1) is a q-analogue of the hypergeometric series ${}_2F_1(\alpha, \beta; \gamma, z)$[3];

$$ {}_2F_1(\alpha, \beta; \gamma, z) := \sum_{n\geq 0} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n. \tag{2}$$

This series (2) has the following famous degeneration diagram

$$\begin{tikzcd}
 & & \text{Weber} \arrow[dr] & \\
\text{Gauss} \arrow[r] & \text{Kummer} \arrow[ur] \arrow[dr] & & \text{Airy} \\
 & & \text{Bessel} \arrow[ur] &
\end{tikzcd} \tag{3}$$

Recently, Y. Ohyama [11] shows that there exists "the digeneration diagram" of Heine's series (1) as follows:

$$\begin{tikzcd}
 & & J_\nu^{(3)} \arrow[r] & q\text{-Airy} \\
{}_2\varphi_1(a,b;c;z) \arrow[r] & {}_1\varphi_1(a;c;z) \arrow[ur] \arrow[r] \arrow[dr] & J_\nu^{(1)}, J_\nu^{(2)} \arrow[r] & \text{Ramanujan} \\
 & & {}_1\varphi_1(a;0;z) \arrow[ur] &
\end{tikzcd}$$

We remark that there exist three different q-Bessel functions $J_\nu^{(j)}, j = 1, 2, 3$[2] and two q-analogues of the Airy function. In this point, this diagram is essentially different from the diagram (3).

Ismail has pointed out that the Ramanujan function is one of q-analogues of the Airy function [6]. The Ramanujan function appears in the third identity on p.57 of Ramanujan's "Lost notebook" [12] as follows (with x replaced by q):

$$\mathrm{A}_q(-a) = \sum_{n\geq 0} \frac{a^n q^{n^2}}{(q;q)_n} = \prod_{n\geq 1} \left(1 + \frac{aq^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \cdots} \right)$$

where

$$
\begin{aligned}
y_1 &= \frac{1}{(1-q)\psi^2(q)}, \\
y_2 &= 0, \\
y_3 &= \frac{q+q^3}{(1-q)(1-q^2)(1-q^3)\psi^2(q)} - \frac{\sum_{n\geq 0}\frac{(2n+1)q^{2n+1}}{1-q^{2n+1}}}{(1-q)^3\psi^6(q)}, \\
y_4 &= y_1 y_3, \\
\psi(q) &= \sum_{n\geq 0} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}.
\end{aligned}
$$

To be precise, the Ramanujan function is given by

$$
A_q(x) := \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}(-x)^n.
$$

This function satisfies the following second order linear q-difference equation;

$$
qxu(q^2x) - u(qx) + u(x) = 0. \tag{4}
$$

The equation (4) has another solution which is given by a divergent series

$$
\theta_q(x)_2\varphi_0\left(0,0;-;q,-\frac{x}{q}\right) = \theta_q(x)\sum_{n\geq 0}\frac{1}{(q;q)_n}\left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{-1}\left(-\frac{x}{q}\right)^n.
$$

Here, $\theta_q(\cdot)$ is the theta function of Jacobi (see the section 2).

An asymptotic formula for the Ramanujan function is obtained by M. E. H. Ismail and C. Zhang as follows[7];

$$
\mathrm{A}_q(x) = \frac{(qx,q/x;q^2)_\infty}{(q;q^2)_\infty}{}_1\varphi_1\left(0;q;q^2,\frac{q^2}{x}\right) - \frac{q(q^2x,1/x;q^2)_\infty}{(1-q)(q;q^2)_\infty}{}_1\varphi_1\left(0;q^3;q^2,\frac{q^3}{x}\right). \tag{5}
$$

From the viewpoint of connection problems on q-difference equations, we can regard the formula (5) as one of connection formulae of the Ramanujan function.

The other q-analogue of the Airy function is known as the q-Airy function $\mathrm{Ai}_q(\cdot)$. The q-Airy function has found in the study of the second q-Painlevé equation[4]. The function $\mathrm{Ai}_q(\cdot)$ is defined by

$$\mathrm{Ai}_q(x) := \sum_{n\geq 0} \frac{1}{(-q;q)_n(q;q)_n} \left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\} (-x)^n$$

and satisfies the following q-difference equation

$$u(q^2x) + xu(qx) - u(x) = 0. \tag{6}$$

The other solution of the equation (6) around the origin is given by

$$u(x) = \frac{\theta_q(q^2x)}{\theta_q(-q^2x)} \mathrm{Ai}_q(-x).$$

Ismail also has pointed out the Ramanujan function and the q-Airy function are different. But the relation between them has not known. In the section 3, we give the connection formula between these functions with using the q-Borel-Laplace transformations of the second kind.

Theorem *For any* $x \in \mathbb{C}^*$, we have

$$\mathrm{A}_{q^2}\left(-\frac{q^3}{x^2}\right) = \frac{1}{(q,-1;q)_\infty}\left\{\theta\left(\frac{x}{q}\right)\mathrm{Ai}_q(-x) + \theta\left(-\frac{x}{q}\right)\mathrm{Ai}_q(x)\right\}.$$

Connection problems on linear q-difference equations between the origin and the infinity are studied by G. D. Birkhoff [1]. The first example of the connection formula was found by G. N. Watson [14] in 1912. This formula is known as "Watson's formula for ${}_2\varphi_1(a,b;c;q,x)$" as follows [2];

$$\begin{aligned}{}_2\varphi_1\left(a,b;c;q;x\right) = &\frac{(b,c/a;q)_\infty(ax,q/ax;q)_\infty}{(c,b/a;q)_\infty(x,q/x;q)_\infty}{}_2\varphi_1\left(a,aq/c;aq/b;q;cq/abx\right)\\ &+\frac{(a,c/b;q)_\infty(bx,q/bx;q)_\infty}{(c,a/b;q)_\infty(x,q/x;q)_\infty}{}_2\varphi_1\left(b,bq/c;bq/a;q;cq/abx\right).\end{aligned} \tag{7}$$

But other connection formulae had not found for a long time. Recently, C. Zhang gives connection formulae for some confluent type basic hypergeometric series [15, 16, 17]. In [16], Zhang gives a connection formula of Jackson's first and second q-Bessel function $J_\nu^{(j)}(x;q), (j=1,2)$;

$$J_\nu^{(1)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty}\left(\frac{x}{2}\right)^\nu \sum_{n\geq 0}\frac{1}{(q^{\nu+1};q)_n}\left(-\frac{x^2}{4}\right)^n$$

and

$$J_\nu^{(2)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} \left(\frac{x}{2}\right)^\nu \sum_{n\geq 0} \frac{q^{n^2}}{(q^{\nu+1};q)_n} \left(-\frac{q^\nu x^2}{4}\right)^n$$

with using the q-Borel-Laplace transformations of the second kind $\mathcal{B}_q^-$ and $\mathcal{L}_q^-$. These transformations are defined for a formal power series $f(x) = \sum_{n\geq 0} a_n x^n$, $a_0 = 1$ as follow;

1. The q-Borel transformation of the second kind is

$$(\mathcal{B}_q^- f)(\xi) := \sum_{n\geq 0} a_n q^{-\frac{n(n-1)}{2}} \xi^n \,(=: g(\xi))\,.$$

2. The q-Laplace transformation of the second kind is

$$\left(\mathcal{L}_q^- g\right)(x) := \frac{1}{2\pi i} \int_{|\xi|=r} g(\xi)\theta_q\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi},$$

where $r > 0$ is enough small number.

In [9] and [10], we obtained connection formulae of the Hahn-Exton q-Bessel function

$$J_\nu^{(3)}(x;q) := \frac{(q^{\nu+1};q)_\infty}{(q;q)_\infty} x^\nu \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(q^{\nu+1};q)_n} \left(-x^2\right)^n$$

and the q-confluent type basic hypergeometric function

$${}_1\varphi_1(a;b;q,x) := \sum_{n\geq 0} \frac{(a;q)_n}{(b;q)_n(q;q)_n} (-1)^n q^{\frac{n(n-1)}{2}} x^n$$

by these transformations. In section 3, we use these transformations to obtain connection formula between the Ramanujan function and the q-Airy function.

On the other hand, the q-Borel-Laplace transformations of the first kind are defined for a formal power series as follow;

1. The q-Borel transformation of the first kind is

$$\left(\mathcal{B}_q^+ f\right)(\xi) := \sum_{n\geq 0} a_n q^{\frac{n(n-1)}{2}} \xi^n \,(=: \varphi(\xi))\,.$$

2. The q-Laplace transformation of the first kind is

$$\left(\mathcal{L}_q^+\varphi\right)(x) := \frac{1}{1-q}\int_0^{\lambda\infty}\frac{\varphi(\xi)}{\theta_q\left(\frac{\xi}{x}\right)}\frac{d_q\xi}{\xi} = \sum_{n\in\mathbb{Z}}\frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)},$$

here, this transformation is given by Jackson's q-integral [2].

These two different types of q-Borel-Laplace transformations are introduced by J. Sauloy [13] and stusied by C. Zhang. We remark that each q-Borel transformation is formal inverse of each q-Laplace transformation, i.e.,

$$\mathcal{L}_q^\pm \circ \mathcal{B}_q^\pm f = f.$$

The application of the q-Borel-Laplace transformations of the first kind is found in [15, 17]. Zhang gives the connection formula of the divergent basic hypergeometric series ${}_2\varphi_0(a,b;-;q,x)$ as follows;
Theorem (Zhang, [15]) *For any $x \in \mathbb{C}^*$, we have*

$$\begin{aligned}&{}_2f_0(a,b;\lambda,q,x)\\ &=\frac{(b;q)_\infty}{(b/a;q)_\infty}\frac{\theta_q(a\lambda)}{\theta_q(\lambda)}\frac{\theta_q(qax/\lambda)}{\theta_q(\lambda/x)}{}_2\varphi_1\left(a,0;\frac{aq}{b};q,\frac{q}{abx}\right)\\ &+\frac{(a;q)_\infty}{(a/b;q)_\infty}\frac{\theta_q(b\lambda)}{\theta_q(\lambda)}\frac{\theta_q(qbx/\lambda)}{\theta_q(\lambda/x)}{}_2\varphi_1\left(b,0;\frac{bq}{a};q,\frac{q}{abx}\right)\end{aligned}$$

where $\lambda \in \mathbb{C}^* \setminus \{-q^n; n \in \mathbb{Z}\}$.

Here, ${}_2f_0(a,b;\lambda,q,x)$ in the left-hand side is the q-Borel-Laplace transform of the function ${}_2\varphi_0(a,b;-;q,x)$. But other application of this method (of the first kind) has not known. In the section 4, we show the connection formula of the divergent series

$${}_2\varphi_0(a,b;-;q,x) = \sum_{n\geq 0}\frac{1}{(q;q)_n}\left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{-1}x^n.$$

This formula is given by the following theorem;
Theorem *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$,*

$$\begin{aligned}\theta_q(x){}_2f_0\left(0,0;-;q,-\frac{x}{q}\right) &= (q;q)_\infty\frac{\theta_q(x)\theta_{q^2}\left(-\frac{\lambda^2}{qx}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)}{}_1\varphi_1\left(0;q;q^2,\frac{q^2}{x}\right)\\ &+\frac{(q;q)_\infty}{1-q}\frac{\theta_q(x)\theta_{q^2}\left(-\frac{\lambda^2}{x}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)}\frac{\lambda}{x}{}_1\varphi_1\left(0;q^3;q^2,\frac{q^3}{x}\right).\end{aligned}$$

2 Basic notations

In this section, we review our notations. We assume that $q \in \mathbb{C}^*$ satisfies $0 < |q| < 1$. The q-shifted operator σ_q is given by $\sigma_q f(x) = f(qx)$. For any fixed $\lambda \in \mathbb{C}^* \setminus q^{\mathbb{Z}}$, the set $[\lambda; q]$-spiral is $[\lambda; q] := \lambda q^{\mathbb{Z}} = \{\lambda q^k; k \in \mathbb{Z}\}$. The (generalized) basic hypergeometric series ${}_r\varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x)$ is

$$
\begin{aligned}
&{}_r\varphi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, x) \\
&\qquad := \sum_{n \geq 0} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n (q; q)_n} \left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{1+s-r} x^n.
\end{aligned}
$$

This series has radius of convergence $\infty, 1$ or 0 according to whether $r - s < 1, r - s = 1$ or $r - s > 1$ (see [2] for further details). In connection problems, the theta function of Jacobi is important. This function is defined by

$$
\theta_q(x) := \sum_{n \in \mathbb{Z}} q^{\frac{n(n-1)}{2}} x^n, \qquad x \in \mathbb{C}^*.
$$

We denote $\theta_q(\cdot)$ or more shortly $\theta(\cdot)$. The theta function has the following properties;

1. Jacobi's triple product identity is
$$
\theta_q(x) = \left(q, -x, -\frac{q}{x}; q\right)_\infty.
$$

2. The q-difference equation which the theta function satisfies;
$$
\theta_q(q^k x) = q^{-\frac{n(n-1)}{2}} x^{-k} \theta_q(x), \quad \forall k \in \mathbb{Z}.
$$

3. The inversion formula;
$$
\theta_q\left(\frac{1}{x}\right) = \frac{1}{x}\theta_q(x).
$$

We remark that the function $\theta(-\lambda x)/\theta(\lambda x)$, $\lambda \in \mathbb{C}^*$ satisfies a q-difference equation
$$
u(qx) = -u(x)
$$
which is also satisfied by the function $u(x) = e^{\pi i \left(\frac{\log x}{\log q}\right)}$.

3 Two types of the q-analogue of the Airy function and the connection formula

There are two different q-analogue of the Airy function. One is called the Ramanujan function which appears in [12]. Ismail [6] pointed out that the Ramanujan function can be considered as a q-analogue of the Airy function. The other one is called the q-Airy function which is obtained by K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada [8] . In this section, we see the properties of these functions. We explain the reason why they are called q-analogue of the Airy function and we show q-difference equations which they satisfy.

3.1 The Ramanujan function $\mathrm{A}_q(x)$

The Ramanujan function appears in Ramanujan's "Lost notebook" [12]. Ismail has pointed out that the Ramanujan function can be considered as a q-analogue of the Airy function. The Ramanujan function is defined by following convergent series;

$$\mathrm{A}_q(x) := \sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}(-x)^n = {}_0\varphi_1(-;0;q,-qx).$$

In the theory of ordinary differencial equations, the term Plancherel-Rotach asymptotics refers to asymptotics around the largest and smallest zeros. With $x = \sqrt{2n+1} - 2^{\frac{1}{2}}3^{\frac{1}{3}}n^{\frac{1}{6}}t$ and for $t \in \mathbb{C}$, the Plancherel-Rotach asymptotic formula for Hermite polynomials $H_n(x)$ is

$$\lim_{n\to+\infty} \frac{e^{-\frac{x^2}{2}}}{3^{\frac{1}{3}}\pi^{-\frac{3}{4}}2^{\frac{n}{2}+\frac{1}{4}}\sqrt{n!}} H_n(x) = \mathrm{Ai}(t). \tag{8}$$

In [6], Ismail shows the q-analogue of (8);

Proposition 1. *One can get*

$$\lim_{n\to\infty} \frac{q^{n^2}}{t^n} h_n(\sinh \xi_n|q) = \mathrm{A}_q\left(\frac{1}{t^2}\right)$$

where $e^{\xi_n} = tq^{-\frac{n}{2}}$.

Here, $h_n(\cdot|q)$ is the q-Hermite polynomial. In this sense, we can deal with the Ramanujan function $\mathrm{A}_q(x)$ as a q-analogue of the Airy function. The Ramanujan function satisfies the following q-diference equation;

$$\left(qx\sigma_q^2 - \sigma_q + 1\right) u(x) = 0. \tag{9}$$

Remark 1. *We remark that another solution of the equation (9) is given by*

$$u(x) = \theta(x)_2\varphi_0(0, 0; -; q, -x/q).$$

Here,

$${}_2\varphi_0\left(0, 0; -; q, -\frac{x}{q}\right) = \sum_{n\geq 0} \frac{1}{(q;q)_n} \left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\}^{-1} \left(-\frac{x}{q}\right)^n$$

is a divergent series.

3.2 The q-Airy function $\mathrm{Ai}_q(x)$

The q-Airy function is found by K. Kajiwara, T. Masuda, M. Noumi, Y. Ohta and Y. Yamada [8], in their study of the q-Painlevé equations. This function is the special solution of the second q-Painlevé equations and given by the following series

$$\mathrm{Ai}_q(x) := \sum_{n\geq 0} \frac{1}{(-q, q; q)_n} \left\{(-1)^n q^{\frac{n(n-1)}{2}}\right\} (-x)^n = {}_1\varphi_1(0; -q; q, -x).$$

T. Hamamoto, K. Kajiwara, N. S. Witte [4] proved following asymptotic expansions;

Proposition 2. *With $q = e^{-\frac{\delta^3}{2}}$, $x = -2ie^{-\frac{s}{2}\delta^2}$ as $\delta \to 0$,*

$${}_1\varphi_1(0; -q; q, -qx) = 2\pi^{\frac{1}{2}}\delta^{-\frac{1}{2}}e^{-\left(\frac{\pi i}{\delta^3}\right)\ln 2 + \left(\frac{\pi i}{2\delta}\right)s + \frac{\pi i}{12}} \left[\mathrm{Ai}\left(se^{\frac{\pi i}{3}}\right) + O(\delta^2)\right],$$

$${}_1\varphi_1(0; -q; q, qx) = 2\pi^{\frac{1}{2}}\delta^{-\frac{1}{2}}e^{-\left(\frac{\pi i}{\delta^3}\right)\ln 2 - \left(\frac{\pi i}{2\delta}\right)s - \frac{\pi i}{12}} \left[\mathrm{Ai}\left(se^{-\frac{\pi i}{3}}\right) + O(\delta^2)\right]$$

for s in any compact domain of $\mathbb{C}$.

Here, $\mathrm{Ai}(\cdot)$ is the Airy function. From this proposition, we can regard the q-Airy function as a q-analogue of the Airy function.

We can easily check out that the q-Airy function satisfies the second order linear q-difference equation

$$\left(\sigma_q^2 + x\sigma_q - 1\right) u(x) = 0. \tag{10}$$

Another solution of the equation (10) is given by

$$u(x) = e^{\pi i\left(\frac{\log x}{\log q}\right)} {}_1\varphi_1(0; -q; q, x) = e^{\pi i\left(\frac{\log x}{\log q}\right)} \mathrm{Ai}_q(-x).$$

3.3 Covering transformations

We define a covering transformation of a second order linear q-difference equation.

Definition 1. *For a q-difference equation*

$$a(x)u(q^2x) + b(x)u(qx) + c(x)u(x) = 0, \tag{11}$$

we define the covering transformation as follows

$$t^2 := x, \quad v(t) := u(t^2), \quad p := \sqrt{q}.$$

The covering transform of the equation (11) is given by

$$a(t^2)v(p^2t) + b(t^2)v(pt) + c(t^2)v(t) = 0.$$

By the covering transformation, the equation

$$\left(K \cdot x\sigma_q^2 - \sigma_q + 1\right) u(x) = 0$$

is transformed to

$$\left(K \cdot t^2\sigma_p^2 - \sigma_p + 1\right) v(t) = 0, \tag{12}$$

where K is a fixed constant in $\mathbb{C}^*$.

3.4 The q-Airy equation around the infinity

We consider the behavior of the equation (10) around the infinity. We set $x = 1/t$ and $z(t) = u(1/t)$. Then $z(t)$ satisfies

$$\left(-\sigma_q^2 + \frac{1}{q^2 t}\sigma_q + 1\right) z(t) = 0.$$

We set $\mathcal{E}(t) = 1/\theta(-q^2 t)$ and $f(t) = \sum_{n\geq 0} a_n t^n, \quad a_0 = 1$. We assume that $z(t)$ can be described as

$$z(t) = \mathcal{E}(t) f(t) = \frac{1}{\theta(-q^2 t)} \left(\sum_{n\geq 0} a_n t^n\right).$$

The function $\mathcal{E}(t)$ has the following property;

Lemma 1. *For any $t \in \mathbb{C}^*$,*

$$\sigma_q \mathcal{E}(t) = -q^2 t \mathcal{E}(t), \qquad \sigma_q^2 \mathcal{E}(t) = q^5 t^2 \mathcal{E}(t).$$

From this lemma, $f(t)$ satisfies the following equation

$$\left(-q^5 t^2 \sigma_q^2 - \sigma_q + 1\right) f(t) = 0. \tag{13}$$

Since (13) is the same as (12) for $K = -q^5$, we obtain

$$f(t) = {}_0\varphi_1(-; 0; q^2, q^5 t^2) = \mathrm{A}_{q^2}(-q^3 t^2).$$

We show a connection formula for $f(t)$. In order to obtain a connection formula, we need the q-Borel transformation and the q-Laplace transformation following Zhang [16].

3.5 The q-Borel transformation and the q-Laplace transformation

Definition 2. *For $f(t) = \sum_{n\geq 0} a_n t^n$, the q-Borel transformation is defined by*

$$g(\tau) = \left(\mathcal{B}_q^- f\right)(\tau) := \sum_{n\geq 0} a_n q^{-\frac{n(n-1)}{2}} \tau^n,$$

and the q-Laplace transformation is given by

$$\left(\mathcal{L}_q^- g\right)(t) := \frac{1}{2\pi i} \int_{|\tau|=r} g(\tau) \theta\left(\frac{t}{\tau}\right) \frac{d\tau}{\tau}, \qquad 0 < r < \frac{1}{|q^2|}.$$

The q-Borel transformation can be considered as a formal inverse of the q-Laplace transformation.

Lemma 2. *For any entire function f,*

$$\mathcal{L}_q^- \circ \mathcal{B}_q^- f = f.$$

Proof. We can prove this lemma calculating residues of the q-Laplace transformation around the origin. $\square$

The q-Borel transformation has following operational relation;

Lemma 3. *For any $l, m \in \mathbb{Z}_{\geq 0}$,*

$$\mathcal{B}_q^-(t^m \sigma_q^l) = q^{-\frac{m(m-1)}{2}} \tau^m \sigma_q^{l-m} \mathcal{B}_q^-.$$

3.6 The connection formula of the q-Airy function

Applying the q-Borel transformation in 3.5 to the equation (12) and using lemma 3, we obtain the first order q-difference equation

$$g(q\tau) = (1 + q^2\tau)(1 - q^2\tau)g(\tau).$$

Since $g(0) = 1$, $g(\tau)$ is given by an infinite product

$$g(\tau) = \frac{1}{(-q^2\tau; q)_\infty (q^2\tau; q)_\infty}$$

which has single poles at

$$\left\{\tau; \tau = \pm q^{-2-k}, \quad \forall k \in \mathbb{Z}_{\geq 0}\right\}.$$

By Cauchy's residue theorem, the q-Laplace transform of $g(\tau)$ is

$$\begin{aligned} f(t) =& \frac{1}{2\pi i} \int_{|\tau|=r} g(\tau)\theta\left(\frac{t}{\tau}\right)\frac{d\tau}{\tau} \\ =& -\sum_{k\geq 0} \mathrm{Res}\left\{g(\tau)\theta\left(\frac{t}{\tau}\right)\frac{1}{\tau}; \tau = -q^{-2-k}\right\} \\ & -\sum_{k\geq 0} \mathrm{Res}\left\{g(\tau)\theta\left(\frac{t}{\tau}\right)\frac{1}{\tau}; \tau = q^{-2-k}\right\} \end{aligned}$$

where $0 < r < r_0 := 1/|q^2|$. We can culculate the residue from lemma 4.

Lemma 4. *For any $k \in \mathbb{N}$, $\lambda \in \mathbb{C}^*$, one can get;*

1. $\mathrm{Res}\left\{\frac{1}{(\tau/\lambda;q)_\infty}\frac{1}{\tau} : \tau = \lambda q^{-k}\right\} = \frac{(-1)^{k+1} q^{\frac{k(k+1)}{2}}}{(q;q)_k (q;q)_\infty}$,

2. $\frac{1}{(\lambda q^{-k};q)_\infty} = \frac{(-\lambda)^{-k} q^{\frac{k(k+1)}{2}}}{(\lambda;q)_\infty (q/\lambda;q)_k}, \quad \lambda \notin q^{\mathbb{Z}}.$

Summing up all of residues, we obtain

$$f(t) = \frac{\theta(q^2t)}{(q,-1;q)_\infty} {}_1\varphi_1\left(0,-q;q,\frac{1}{t}\right) + \frac{\theta(-q^2t)}{(q,-1;q)_\infty} {}_1\varphi_1\left(0,-q;q,-\frac{1}{t}\right).$$

We obtain a connection formula for $z(t) = \mathcal{E}(t)f(t)$. Finally, we acquire the following connection formula between the Ramanujan function and the q-Airy function.

Theorem 1. *For any $x \in \mathbb{C}^*$,*

$$\mathrm{A}_{q^2}\left(-\frac{q^3}{x^2}\right) = \frac{1}{(q,-1;q)_\infty}\left\{\theta\left(\frac{x}{q}\right)\mathrm{Ai}_q(-x) + \theta\left(-\frac{x}{q}\right)\mathrm{Ai}_q(x)\right\}.$$

Here, both $\mathrm{A}_q(x)$ and $\mathrm{Ai}_q(x)$ are defined by convergent series on whole of the complex plain. The connection formula above is valid for any $x \in \mathbb{C}^*$.

4 Connection formula of the divergent series ${}_2\varphi_0(0,0;-;q,\cdot)$

In this section, we show a connection formula of the divergent series ${}_2\varphi_0$. This series appears in the second solution of the Ramanujan equation (9). At first, we review two q-exponential functions to consider our connection formula.

4.1 Two different q-exponential functions

In this section, we review two different q-exponential functions from the viewpoint of the connection problems. One of the q-exponential function $e_q(x)$ is given by

$$e_q(x) := {}_1\varphi_0(0;-;q,x) = \sum_{n\geq 0}\frac{x^n}{(q;q)_n}.$$

The other q-exponential function $E_q(x)$ is

$$E_q(x) := {}_0\varphi_0(-;-;q,-x) = \sum_{n\geq 0} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} x^n.$$

The function $e_q(x)$ satisfies the following first order q-difference equation

$$\{\sigma_q - (1-x)\}\, u(x) = 0$$

and $E_q(x)$ satisfies

$$\{(1+x)\sigma_q - 1\}\, u(x) = 0.$$

The limit $q \to 1-0$ converges the exponential function

$$\lim_{q\to 1-0} e_q\left(x(1-q)\right) = \lim_{q\to 1-0} E_q\left(x(1-q)\right) = e^x.$$

In this sense, these functions considered as q-analogues of the exponential function. It is known that there exists the relation between these functions:

$$e_q(x)E_q(-x) = 1, \quad e_{q^{-1}}(x) = E_q(-qx).$$

But another relation has not known. We show the connection formula between them and give alternate representation of $e_q(\cdot)$.

4.2 The connection formula and alternate representation

At first, we show the following connection formula between $e_q(\cdot)$ and $E_q(\cdot)$.

Theorem 2. *For any $x \in \mathbb{C}^* \setminus [1;q]$,*

$$e_q(x) = \frac{(q;q)_\infty}{\theta_q(-x)} E_q\left(-\frac{q}{x}\right)$$

where $|x| < 1$.

Proof. The function $e_q(x)$ and $E_q(x)$ have infinite product as follows:

$$e_q(x) = \frac{1}{(x;q)_\infty}, \qquad |x| < 1$$

and

$$E_q(x) = (-x;q)_\infty.$$

We remark that $e_q(x)$ can be described as

$$e_q(x) = \frac{1}{\theta_q(-x)}\left(q, \frac{q}{x};q\right)_\infty = \frac{(q;q)_\infty}{\theta_q(-x)} E_q\left(-\frac{q}{x}\right)$$

where $|x| < 1$. We obtain the conclusion. $\square$

Therefore, these q-exponential functions are related by the connection formula between the origin and the infinity. If we replace x by x/q, we obtain the following lemma. This is useful to consider the connection problem in the last section.

Lemma 5. *For any $x \in \mathbb{C}^* \setminus [1;q]$, the function $e_q(x/q)$ has the following alternate representation.*

$$e_q\left(\frac{x}{q}\right) = \frac{(q;q)_\infty}{\theta_q\left(-\frac{x}{q}\right)} {}_0\varphi_1\left(-;q;q^2, \frac{q^5}{x^2}\right) - \frac{(q;q)_\infty}{\theta_q\left(-\frac{x}{q}\right)} \frac{q^2}{(1-q)x} {}_0\varphi_1\left(-;q^3;q^2, \frac{q^7}{x^2}\right).$$

Proof. From theorem 2,

$${}_1\varphi_0\left(0;-;q,\frac{x}{q}\right) = \frac{(q;q)_\infty}{\theta_q\left(-\frac{x}{q}\right)} E_q\left(-\frac{q^2}{x}\right) = \frac{(q;q)_\infty}{\theta_q\left(-\frac{x}{q}\right)} {}_0\varphi_0\left(-;-;q,\frac{q^2}{x}\right).$$

Here,

$${}_0\varphi_0\left(-;-;q,\frac{q^2}{x}\right) = \sum_{k\geq 0} \frac{1}{(q;q)_k}(-1)^k q^{\frac{k(k-1)}{2}} \left(\frac{q^2}{x}\right)^k$$

and we remark that $(a;q)_{2k} = (a, aq;q^2)_k$[2]. By separating the terms with even and odd $k \geq 0$, we obtain the conclusion. $\square$

4.3 The connection formula of the series ${}_2\varphi_0(0,0;-;q,\cdot)$

The aim of this section is to give a proof for the following theorem;

Theorem 3. *For any $x \in \mathbb{C}^* \setminus [-\lambda; q]$,*

$$\theta_q(x){}_2f_0\left(0,0;-;q,-\frac{x}{q}\right) = (q;q)_\infty \frac{\theta_q(x)\theta_{q^2}\left(-\frac{\lambda^2}{qx}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)}{}_1\varphi_1\left(0;q;q^2,\frac{q^2}{x}\right)$$
$$+\frac{(q;q)_\infty}{1-q}\frac{\theta_q(x)\theta_{q^2}\left(-\frac{\lambda^2}{x}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)}\frac{\lambda}{x}{}_1\varphi_1\left(0;q^3;q^2,\frac{q^3}{x}\right).$$

We define the q-Borel-Laplace transformations of the first kind to obtain the connection formula between the origin and the infinity.

Definition 3. *For any analytic function f(x), the q-Borel transformation of the first kind $\mathcal{B}_q^+$ is*

$$\left(\mathcal{B}_q^+ f\right)(\xi) := \sum_{n\geq 0} a_n q^{\frac{n(n-1)}{2}}\xi^n =: \varphi(\xi),$$

the q-Laplace transformation of the first kind $\mathcal{L}_q^+$ is

$$\left(\mathcal{L}_q^+\varphi\right)(x) := \sum_{n\in\mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)}.$$

We remark that the q-Borel transformation $\mathcal{B}_q^+$ is formal inverse of the q-Laplace transformation $\mathcal{L}_q^+$ as follows;

Lemma 6. *For any entire function $f(x)$, we have*

$$\mathcal{L}_q^+ \circ \mathcal{B}_q^+ f = f.$$

We give the proof of theorem 3.

Proof. We apply the q-Borel transformation $\mathcal{B}_q^+$ to the divergent series $v(x) = {}_2\varphi_0(0,0;-;q,-x/q)$. We obtain

$$\left(\mathcal{B}_q^+ v\right)(\xi) = {}_1\varphi_0\left(0;-;q,\frac{\xi}{q}\right) =: \varphi(\xi).$$

From lemma 5,

$$\varphi(\xi) = \frac{(q;q)_\infty}{\theta_q\left(-\frac{\xi}{q}\right)}{}_0\varphi_1\left(-;q;q^2,\frac{q^5}{\xi^2}\right) - \frac{(q;q)_\infty}{\theta_q\left(-\frac{\xi}{q}\right)}\frac{q^2}{(1-q)\xi}{}_0\varphi_1\left(-;q^3;q^2,\frac{q^7}{\xi^2}\right)$$

where $|\xi/q| < 1$.

We apply the q-Laplace transformation $\mathcal{L}_q^+$ to $\varphi(\xi)$:

$$
\begin{aligned}
\left(\mathcal{L}_q^+\varphi\right)(x) &= \sum_{n\in\mathbb{Z}} \frac{\varphi(\lambda q^n)}{\theta_q\left(\frac{\lambda q^n}{x}\right)} = \sum_{n\in\mathbb{Z}} \frac{{}_1\varphi_0\left(0; -; q, \frac{\lambda q^n}{q}\right)}{\theta_q\left(\frac{\lambda q^n}{x}\right)} \\
&= \frac{(q;q)_\infty}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} \sum_{n-m\in\mathbb{Z}} (q^2)^{\frac{(n-m)(n-m-1)}{2}} \left(-\frac{\lambda^2}{qx}\right)^{n-m} \\
&\qquad \times \sum_{m\geq 0} \frac{(-1)^m (q^2)^{\frac{m(m-1)}{2}}}{(q;q^2;q^2)_m} \left(\frac{q^2}{x}\right)^m \\
&- \frac{(q;q)_\infty}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} \frac{q^2}{(1-q)\lambda} \sum_{n-m\in\mathbb{Z}} (q^2)^{\frac{(n-m)(n-m-1)}{2}} \left(-\frac{\lambda^2}{q^2x}\right)^{n-m} \\
&\qquad \times \sum_{m\geq 0} \frac{(-1)^m (q^2)^{\frac{m(m-1)}{2}}}{(q^3,q^2;q^2)_m} \left(\frac{q^3}{x}\right)^m .
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&{}_2f_0\left(0,0;-;q,-\frac{x}{q}\right) = \mathcal{L}_q^+ \circ \mathcal{B}_q^+ {}_2\varphi_0\left(0,0;-;q,-\frac{x}{q}\right) \\
&= (q;q)_\infty \frac{\theta_{q^2}\left(-\frac{\lambda^2}{qx}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} {}_1\varphi_1\left(0;q;q^2,\frac{q^2}{x}\right) + \frac{(q;q)_\infty}{1-q} \frac{\theta_{q^2}\left(-\frac{\lambda^2}{x}\right)}{\theta_q\left(-\frac{\lambda}{q}\right)\theta_q\left(\frac{\lambda}{x}\right)} {}_1\varphi_1\left(0;q^3;q^2,\frac{q^3}{x}\right).
\end{aligned}
$$

We obtain the conclusion. □

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